PREDICTION INTERVALS FOR CORRELATED SAMPLES

Chang Heup Choi

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

PREDICTION INTERVALS
FOR
CORRELATED SAMPLES

by

Chang Heup Choi

September 1974

Thesis Advisor: T. Jayachandran

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Prediction Intervals for Correlated Samples		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis; September 1974
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Chang Heup Choi		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE September 1974
		13. NUMBER OF PAGES 50
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Postgraduate School Monterey, California 93940		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Prediction Intervals
Correlated Samples

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
(Page 1) S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(20. ABSTRACT Cont'd)

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Chang Heup Choi
Lieutenant Colonel, Republic of Korea Army
B.S., Republic of Korea Military Academy, 1958
B.S., Seoul National University, 1962

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1974

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In this thesis we extend several results on prediction intervals that were obtained under the assumption that the sample observations are independent and identically distributed as $N(\mu,\sigma^2)$. We assume that the samples are correlated with a prescribed correlation structure and show that many of the results available for the independent case apply equally well for the correlated samples. The correlation structure assumed occurs in variance components models in Analysis of Variance and the results can also be applied to the case where the samples have an intraclass correlation (equicorrelated samples).

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I. INTRODUCTION

A prediction interval is a random interval that contains the value of a future observation or some function of future observations and whose end points are functions of previous sample values. Such an interval provides an indication of the uncertainty in the future observations. More specifically, a 100r percent prediction interval for the value of a future sample is an interval that is based on a previous sample and encloses the future observations with probability r, independent of the values of the distribution parameters, such as the mean or the standard deviation. A prediction interval needs to be distinguished both from a confidence interval and a tolerance interval; a confidence interval encloses the value of an unknown parameter and a tolerance interval is an interval within which a specified proportion of the population values will lie with a specified probability.

In many practical problems, it would be of interest to construct a prediction interval for the values of the next k sample values from a population. For example, if only one machine is available for testing and we must perform trials sequentially, a prediction interval could provide helpful information about the total time needed to complete the experiment or perhaps the number of trials it would be possible to perform. Another application of prediction

intervals is in forecasting before a planned experiment is completed. In an experiment where each observation is expensive or where they can be made only infrequently, prediction intervals may be helpful in reaching a decision on the profitability of continuing the experiment at intermediate points in the experiment. For example, when the experiment concerns a physical input or output, preliminary estimates of the ultimate amount of needed input material or of the ultimate storage needed for the output might be helpful. In other situations where the random variable is the "time until occurrence of an event," and where physical limitations prevent the concurrent running of all planned trials, prediction intervals might provide helpful information concerning the total time until completion of the planned experiment. Prediction intervals are also of frequent interest to a typical consumer of one or a small number of units of a given product. Such an individual is generally more directly concerned with the future performance of his specific sample than in the process from which the sample had been selected. A prediction interval to contain each of the values of the sample would then provide him with an interval within which he may expect the performance of all his units to be located with a high probability. Based upon his experience with a previous sample of 10 light bulbs, a consumer might wish to construct an interval which would have a high probability of including the performance values of each of three additional bulbs.

In this thesis we derive prediction intervals for one future sample observation as well as simultaneous intervals for a specified number of future sample observations when the samples are correlated. These results are obtained as extensions of results due to Hahn [5]. He derived similar intervals for the case where the samples are independent and identically distributed as $N(\mu,\sigma^2)$.

In Chapter III it is shown that Hahn's prediction interval for the standard deviation of a single future sample is valid even in the case where the sample values are correlated and have a multivariate normal distribution with mean vector $\underline{\mu} = (\mu,\mu,\mu,\ldots,\mu)'$ and covariance matrix $\underline{V}$ having the following structure:

$$\underset{n\times n}{\underline{V}} = \frac{1}{2}\left(\underset{n\times n}{\underline{H}} + \underset{n\times n}{\underline{H}'}\right) + \alpha\left(\underset{n\times n}{\underline{I}} - \underset{n\times n}{\underline{E}}\right) \tag{1.1}$$

$$\text{where } \underset{n\times n}{\underline{H}} = \begin{pmatrix} h_1 & h_1 & h_1 & \cdots\cdots & h_1 \\ h_2 & h_2 & h_2 & \cdots\cdots & h_2 \\ h_3 & h_3 & h_3 & \cdots\cdots & h_3 \\ \vdots & \vdots & \vdots & & \vdots \\ h_n & h_n & h_n & \cdots\cdots & h_n \end{pmatrix}$$

$\underset{n\times n}{\underline{H}'}$ is the transpose of $\underset{n\times n}{\underline{H}}$, h_i $(i=1,2,3,\ldots,n)$ and α are positive constants, $\underset{n\times n}{\underline{I}}$ is an nxn identity matrix,

and $\underline{E}_{nxn}$ is an nxn matrix all of whose elements are unity.

Simultaneous prediction intervals for the standard deviations of k future samples are also derived and examples illustrating the results are provided.

A covariance matrix with the above structure occurs in the study of random effects models in analysis of variance. If samples are drawn from a normal distribution $N(\mu,\sigma^2)$ and it is assumed that μ itself is normally distributed as $N(\eta,\sigma_\mu^2)$ then it can be shown that the sample values have a multivariate normal distribution with mean vector $\underline{\eta} = (\eta,\eta,\eta,\ldots,\eta)'$ and covariance matrix

$$\underline{V} = \begin{pmatrix} \sigma^2+\sigma_\mu^2 & \sigma_\mu^2 & \sigma_\mu^2 & \cdots & \sigma_\mu^2 \\ \sigma_\mu^2 & \sigma^2+\sigma_\mu^2 & \sigma_\mu^2 & \cdots & \sigma_\mu^2 \\ \sigma_\mu^2 & \sigma_\mu^2 & \sigma^2+\sigma_\mu^2 & \cdots & \sigma_\mu^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_\mu^2 & \sigma_\mu^2 & \sigma_\mu^2 & \cdots & \sigma^2+\sigma_\mu^2 \end{pmatrix}.$$

It can be seen that the matrix $\underline{V}$ has the same structure as in (1.1) by letting $h_1=h_2=\ldots=h_n=\sigma^2+\sigma_\mu^2$ and $\alpha=\sigma^2$. A possible application of the results of this thesis is in the following situation. From a lot containing a large number of guns n are selected at random. Each of these guns is then fired k times and the resulting miss distances

from a target are measured. Based on the mean of the measured miss distances, a prediction interval for the miss distance for a randomly chosen gun may be predicted.

Chapter IV deals with procedures for constructing a prediction interval to contain a single additional observation and also with constructing a simultaneous prediction interval to contain all k additional future observations, when the samples are correlated and the covariance matrix has the structure as in equation (1.1).

II. SUMMARY OF KNOWN RESULTS

A. DEFINITIONS AND NOTATIONS

Let X_{ij}, $i=0,1,2,3,\ldots,k$ and $j=1,2,3,\ldots,n_i$, be k+1 sets of random samples of size n_i from a normal distribution $N(\mu,\sigma^2)$. The n_o samples for i=0 are considered as the given sample and the remaining k sets are future samples for which prediction intervals are needed.

Let

$$\overline{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$$

and

$$S_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2$$

where $i=0,1,2,3\ldots,k$ and $j=1,2,3,\ldots,n_i$.

B. PREDICTION INTERVALS FOR THE STANDARD DEVIATIONS OF FUTURE SAMPLES

It is well known that $n_oS_o{}^2/\sigma^2$ and $n_iS_i{}^2/\sigma^2$ $(i=1,2,3\ldots,k)$ have a Chi-square distribution with n_o-1 and n_i-1 degree of freedom respectively and they are mutually independent.

Thus, $S_i{}^2/S_o{}^2$ follows an F distribtuion with n_i-1 and n_o-1 degree of freedom respectively, $i=1,2,3,\ldots,k$.

Therefore, a prediction interval to contain the standard deviation S_i of a single future sample of n_i observations is

$$\Pr\{S_oF(n_i-1,n_o-1;(1-r)/2)^{\frac{1}{2}} < S_i < S_oF(n_i-1,n_o-1;(1+r)/2)^{\frac{1}{2}}\} = r \tag{2.1}$$

where $F(n_i-1,n_o-1;(1-r)/2)$ and $F(n_i-1,n_o-1;(1+r)/2)$ are lower and upper 100r% points of F distribution with n_i-1 and n_o-1 degree of freedom respectively.

A two-sided 100r% prediction interval to contain the standard deviation S_i of a single future sample of size n_i is

$$(F(n_i-1,n_o-1;(1-r)/2)^{\frac{1}{2}}S_o,\ F(n_i-1,n_o-1;(1+r)/2)^{\frac{1}{2}}S_o) \tag{2.2}$$

To obtain a simultaneous interval to contain the standard deviations of k future samples assume that $n_i=m$, $i=1,2,3,\dots,k$, and let

$$\max_i \frac{S_i^2}{S_o^2} = W_L(K,m-1,n_o-1)$$

and

$$\min_i \frac{S_i^2}{S_o^2} = W_S(K,m-1,n_o-1)$$

The random variables $W_L(K,m-1,n_o-1)$ and $W_S(K,m-1,n_o-1)$ are known as the studentized largest and studentized smallest Chi-square variates, respectively, in the statistical literature and some tables [1] of the percentage point of their distributions are available. Let $D_U(K,m-1,n_o-1;r)$ and $D_L(K,m-1,n_o-1,1-r)$ denote the upper 100r% and the lower

100(1-r)% points of the distribution of $W_L(K,m-1,n_o-1)$ and $W_S(K,m-1,n_o-1)$, respectively.

Then

$$\Pr\{\max_i S_i^2 \leq D_U(K,m-1,n_o-1;r)S_o^2\} = r$$

and

$$\Pr\{\min_i S_i^2 \geq D_L(K,m-1,n_o-1,r)S_o^2\} = 1-r \qquad (2.3)$$

A simultaneous prediction interval to contain all the standard deviations $S_1,S_2,S_3\ldots,S_k$ is given by

$$(D_U(K,m-1,n_o-1;r)^{\frac{1}{2}}S_o,\ D_L(K,m-1,n_o-1;1-r)^{\frac{1}{2}}S_o)$$

C. PREDICTION INTERVALS FOR THE OBSERVATIONS IN A FUTURE SAMPLE

Let $X_1,X_2,X_3,\ldots,X_n$ be the values of n given samples from a normal distribution $N(\mu,\sigma^2)$ and let $X_{n+1},X_{n+2},X_{n+3},\ldots,X_{n+k}$ be the values of k future independent observations to be drawn from the same distribution. To get a prediction interval to contain a single additional observation X_{n+1}, we proceed as follows;

Let

$$Z_1 = X_{n+1} - \overline{X}_o$$

the expected value of Z_1 is zero and the standard deviation of Z_1 is $\quad Z_1 \sim N(0,\ \sigma^2(1+\frac{1}{n})$

$$\sigma(1+\frac{1}{n})^{\frac{1}{2}}$$

It is easily seen that that the standardized variable

$$Z_1' = \frac{Z_1 - 0}{\sqrt{\sigma^2(1+\frac{1}{n})}} = \frac{X_{n+1} - \overline{X}_o}{\sigma(1+\frac{1}{n})^{\frac{1}{2}}}$$

and

$$S_o^{\;2} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X}_o)^2$$

are independent. Therefore,

$$T = \frac{Z_1'}{\sqrt{\frac{(n-1)S_o^{\;2}}{\sigma^2(n-1)}}} = \frac{X_{n+1} - \overline{X}_o}{S_o(1+\frac{1}{n})^{\frac{1}{2}}}$$

follows a t distribution with n-1 degrees of freedom.

Thus,

$$Pr\{\overline{X}_o + t(n-1;(1-r)/2)(1+1/n)^{\frac{1}{2}} S_o < X_{n+1} < \overline{X}_o + t(n-1;(1+r)/2)(1+1/n)^{\frac{1}{2}} S\} = r \tag{2.4}$$

where t(n-1;(1-r)/2) and t(n-1;(1+r)/2) are lower and upper 100r% points of t-distribution with n-1 degrees of freedom.

Hence, a two-sided 100r% prediction interval to contain a single future observation X_{n+1} is

$$\bar{X}_o \pm t(n-1);(1+r)/2)(1+1/n)^{\frac{1}{2}}S_o$$

To determine a simultaneous prediction interval to contain all k future observations, first, let

$$Z_i = Z_{n+i} - \bar{X}_o \ , \quad i = 1,2,3,\ldots,k.$$

Then, the expected value of Z_i is zero and the variance of Z_i is

$$\sigma^2(1+\frac{1}{n})$$

and it can be shown that $\mathrm{cov}(Z_i,Z_j) = \sigma^2/n$ for all i and j, $i \neq j$. The transformed variables

$$Z_i' = \frac{Z_i}{\sigma(1+\frac{1}{n})^{\frac{1}{2}}} = \frac{X_{n+i} - \bar{X}_o}{\sigma(1+\frac{1}{n})^{\frac{1}{2}}} \qquad i = 1,2,3,\ldots,k$$

have standard normal distributions.

Since $(n-1)S_o^2/\sigma^2$ is independent of the Z_i' and has a Chi-square distribution with n-1 degrees of freedom, each of the ratios

$$T_i = \frac{Z_i'}{\sqrt{\frac{(n-1)S_o^2}{(n-1)\sigma^2}}} = \frac{X_{n+i} - \bar{X}_o}{S_o(1+\frac{1}{n})^{\frac{1}{2}}}$$

follows a student's t-distribution with n-1 degrees of freedom and the T_i are correlated. The random variables $T_1, T_2, T_3, \ldots, T_K$ are jointly distributed according to the multivariate generalization of the student's t-distribution with n-1 degrees of freedom. Tables of the percentage points of this distribution are given in [4]. If u is defined as the solution of the integral equation

$$r = \int_{-u}^{u} \int_{-u}^{u} \cdots \int_{-u}^{u} f_{T_1, T_2, T_3, \ldots, T_K} \, dT_1, dT_2, dT_3, \ldots, dT_K$$

where $f_{T_1, T_2, T_3 \ldots, T_K}$ is the joint probability density function of multivariate t-distribution with n-1 degrees of freedom, then

$$\Pr\{\bar{X}_o - u(1+\tfrac{1}{n})^{\frac{1}{2}} S_o < X_{n+1} < \bar{X}_o + u(1+\tfrac{1}{n})^{\frac{1}{2}} S_o, \ldots, \bar{X}_o - u(1+\tfrac{1}{n})^{\frac{1}{2}} S_o < X_{n+k} < \bar{X}_o + u(1+\tfrac{1}{n})^{\frac{1}{2}} S_o\} = r$$

The resulting 100r% simultaneous prediction interval to contain the values $X_{n+1}, X_{n+2}, X_{n+3}, \ldots, X_{n+k}$ of all k additional observations is

$$\bar{X}_o \pm u(1+\tfrac{1}{n})^{\frac{1}{2}} S_o \qquad (2.4)$$

D. SOME THEOREMS USED IN DERIVING THE RESULTS IN THE THESIS

* <u>Theorem 1</u>. If $\underline{X}$ is distributed $N(\underline{\mu}, \sigma^2 \underline{I})$, then $\underline{X}'\underline{AX}/\sigma^2$ is distributed as $\chi^2(K, \lambda)$, where $\lambda = \underline{\mu}'\underline{A\mu}/2\sigma^2$, and k = rank of $\underline{A}$, if and only if $\underline{A}$ is idempotent.

* Theorem 2. If $\underline{X}$ is distributed $N(\underline{\mu},\underline{V})$, then $\underline{X}'\underline{BX}$ is distributed as $\chi^2(k,\lambda)$, where $\lambda = \frac{1}{2}\underline{\mu}'\underline{B\mu}$ and k is the rank of $\underline{B}$, if and only if $\underline{BV}$ is idempotent.

* Theorem 3. If $\underline{X}$ is distributed $N(\underline{\mu},\underline{V})$, then $\underline{X}'\underline{AX}$ and $\underline{X}'\underline{BX}$ are idependent if and only if $\underline{AVB} = 0$.

* Theorem 4. If $\underline{X}$ is distributed $N(\underline{\mu},\underline{V})$, then $\underline{Y} = \underline{C}'\underline{X}$ and $\underline{X}'\underline{AX}$ are independent if and only if $\underline{C}'\underline{VA} = 0$.

* Theorem 5. (Hogg and Craig theorem)

Let $Q = Q_1+Q_2+Q_3+\ldots+Q_{k-1}+Q_k$, where $Q, Q_1, Q_2, Q_3, \ldots, Q_{k-1}$, and Q_k are k+1 random variables that are quadratic forms in the observations of a random sample of size n from a normal distribution $N(\mu,\sigma^2)$. Let Q/σ^2 be $\chi^2(r)$, let Q_i/σ^2 be $\chi^2(r_i)$, $i=1,2,3,\ldots,k-1$, and let Q_k be non-negative. Then the random variables $Q_1, Q_2, Q_3, \ldots, Q_k$ are mutually stochastically independent and, hence, Q_k/σ^2 is $\chi^2\ (r_k = r - \sum_{j=1}^{k-1} r_j)$.

* Theorem 6. (Baldessari theorem). Let $\underset{n\times 1}{\underline{X}}$ be a multivariate normal distribution with mean vector $\underset{n\times 1}{\underline{\mu}}$ and covariance matrix $\underset{n\times n}{\underline{V}}$, i.e., $N(\underline{\mu},\underline{V})$, and $\underline{B}_0, \underline{B}_1, \underline{B}_2, \ldots, \underline{B}_k$ be (nxn) idempotent matrices satisfying

$$\sum_{j=0}^{k} \underline{B}_j = \underset{n\times n}{\underline{I}} - \frac{1}{n} \underset{n\times n}{\underline{E}} .$$

where $\underset{n\times n}{\underline{I}}$ is the (nxn) identity matrix and $\underset{n\times n}{\underline{E}}$ is a (nxn) matrix all of whose elements are unity. Let α be a positive constant. Then, a necessary and sufficient

condition for $\underline{X}'B_jX/\alpha$, $j=1,2,3,\ldots,k$, to be mutually independent and have non-central Chi-square distribution with r_j (r_j = rank of B_j, $j=0,1,2,\ldots,k$) degree of freedom is that the covariance matrix $\underline{V}$ has the following structure

$$\underset{n\times n}{\underline{V}} = \frac{1}{2}\left(\underset{n\times n}{\underline{H}} + \underset{n\times n}{\underline{H}'} \right) + \alpha\left(\underset{n\times n}{\underline{I}} - \underset{n\times n}{\underline{E}} \right)$$

where $\underset{n\times n}{\underline{H}}$, $\underset{n\times n}{\underline{I}}$ and $\underset{n\times n}{\underline{E}}$ are defined on (1.1).

III. PREDICTION INTERVALS TO CONTAIN THE STANDARD DEVIATIONS OF FUTURE SAMPLES - CORRELATED CASE

Hahn [5] derived prediction intervals to contain the standard deviations of future samples of independent and identically distributed random variables from a normal distribution with unknown mean and unknown standard deviation. In this chapter we extend Hahn's results to the case where the samples are correlated and have a special type of covariance structure.

Section A deals with the procedures for constructing a prediction interval to contain the standard deviation of a single future sample of size n_1 observations, based on a given sample of size n_o.

Section B deals with the construction of simultaneous prediction intervals to contain the standard deviations S_i $i = 1,2,3,\ldots,k$ of k future samples of sizes n_i.

Numerical examples are given in Section C.

A. PREDICTION INTERVAL TO CONTAIN THE STANDARD DEVIATION OF A SINGLE FUTURE SAMPLE

Let $X_{01}, X_{02}, X_{03}, \ldots, X_{0n_o}$ be the values of a given sample and $X_{11}, X_{12}, X_{13}, \ldots, X_{1n_1}$, the values of a future sample. It is required to construct a prediction interval for the standard deviation S_1 of the future sample based on the standard deviation S_o of the given sample.

Let

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$$

and

$$S_i^{\,2} = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \quad \text{where } i = 0,1$$

Let

$$\bar{X} = \frac{1}{N} \sum_{i=0}^{1} \sum_{j=1}^{n_i} X_{ij} \quad \text{where } N = n_o + n_1$$

and

$$S^2 = \frac{1}{N} \sum_{i=0}^{1} \sum_{j=1}^{n_i} (X_{ij} - \bar{X})^2$$

denote the sample mean and variance of the combined sample of size $N = n_o + n_1$.

Since $\bar{X}_1 = (N\bar{X} - n_o\bar{X}_o)/n_1$, the sum of squares NS^2 can be partitioned as follows:

$$NS^2 = \sum_{i=0}^{1} \sum_{j=1}^{n_i} (X_{ij}-\bar{X})^2 = \sum_{i=0}^{1} \sum_{j=1}^{n_i} (X_{ij}-\bar{X}_i+\bar{X}_i-\bar{X})^2$$

$$= \sum_{i=0}^{1} \sum_{j=1}^{n_i} (X_{ij}-\bar{X}_i)^2 + n_i(\bar{X}_i-\bar{X})^2$$

$$= n_oS_o^{\,2}+n_1S_1^{\,2}+n_o(\bar{X}_o-\bar{X})^2+n_1\{(N\bar{X}-n_o\bar{X}_o)/n_1-\bar{X}\}^2$$

$$= n_oS_o^{\,2} + n_1S_1^{\,2} + \frac{n_oN}{n_1} (\bar{X}-\bar{X}_o)^2 \qquad (3.1)$$

Expressing the sum of squares in (3.1) as quadratic forms we can write the equation as

$$\underline{X}'\underline{B}_0\underline{X} = \underline{X}'\underline{B}_1\underline{X} + \underline{X}'\underline{B}_2\underline{X} + \underline{X}'\underline{B}_3\underline{X} \tag{3.2}$$

where $\underline{B}_o$, $\underline{B}_1$ and $\underline{B}_2$ are idempotent matrices and

$$\underline{B}_o = \underset{N\text{x}N}{\underline{I}} - N^{-1}\underset{N\text{x}N}{\underline{E}} \quad \text{and} \quad \underline{B}_i = \underset{n_i\text{x}n_i}{\underline{I}} - n_i^{-1}\underset{n_i\text{x}n_i}{\underline{E}}, \; i=0,1.$$

If $X_{01}, X_{02}, X_{03}, \ldots, X_{0n_o}, X_{11}, X_{12}, X_{13}, \ldots, X_{1n_1}$ are independent and have identical normal distributions with mean μ and variance σ^2, then it is known that NS^2/σ^2, $n_oS_o^2/\sigma^2$ and $n_1S_1^2/\sigma^2$ have Chi-square distributions with $N-1$, n_o-1 and n_1-1 degrees of freedom respectively and $n_oN(\bar{X}-\bar{X}_o)^2/n_1$ is non-negative. Thus, Hogg and Craig's theorem (theorem 5) applies to equation (3.1). Therefore, the three quadratic forms on the right hand side of (3.1) are mutually independent and $n_oN(\bar{X}-\bar{X}_o)^2/n_1\sigma^2$ has a Chi-square distribution with 1 degree of freedom. It also follows that the matrix $\underline{B}_3$ in equation (3.2) is also idempotent.

Now, suppose $\underset{N\text{x}1}{\underline{X}} = (X_{01}, X_{02}, X_{03}, \ldots, X_{0n_o}, X_{11}, X_{12}, X_{13}, \ldots, X_{1n_1})'$ is a vector random variable having a multivariate normal distribution with mean $\underset{N\text{x}1}{\underline{\mu}} = (\mu, \mu, \mu, \ldots, \mu)'$ and covariance matrix $\underset{N\text{x}N}{\underline{V}}$ which has the following structure.

$$\underset{N\text{x}N}{\underline{V}} = \tfrac{1}{2}(\underset{N\text{x}N}{\underline{H}} + \underset{N\text{x}N}{\underline{H}'}) + \alpha(\underset{N\text{x}N}{\underline{I}} - \underset{N\text{x}N}{\underline{E}}) \tag{3.3}$$

where

$$\underset{N\times N}{\underline{H}} = \begin{pmatrix} h_1 & h_1 & h_1 & \cdots\cdots\cdots & h_1 \\ h_2 & h_2 & h_2 & \cdots\cdots\cdots & h_2 \\ h_3 & h_3 & h_3 & \cdots\cdots\cdots & h_3 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ h_N & h_N & h_N & \cdots\cdots\cdots & h_N \end{pmatrix}$$

$\underset{N\times N}{\underline{I}}$ is an NXN identity matrix, $\underset{N\times N}{\underline{E}}$ is an NXN matrix whose elements are all unity and α and h_i, $i = 1,2,3,\ldots,N$, are positive constant.

To obtain a prediction interval for S_1, we start with equations (3.1) and (3.2). Since the matrices $\underline{B}_o$, $\underline{B}_1$, $\underline{B}_2$ and $\underline{B}_3$ are idempotent matrices and $\underset{N\times N}{\underline{B}_o} = \sum_{j=1}^{3} \underline{B}_j = \underset{N\times N}{\underline{I}} - N^{-1}\underset{N\times N}{\underline{E}}$ all the conditions of the Baldessari theorem (theorem 6) are now satisfied. Therefore, the three quadratic forms of (3.2) on the right hand side have central Chi-square distributions with n_o-1, n_1-1 and 1 degree of freedom respectively and are mutually independent.

Thus, the random variable

$$\frac{\underline{X}'\underline{B}_2\underline{X}}{\alpha} \Bigg/ \frac{\underline{X}'\underline{B}_1\underline{X}}{\alpha} = \frac{(n_1-1)S_1^{\;2}}{\alpha(n_1-1)} \Bigg/ \frac{(n_o-1)S_o^{\;2}}{\alpha(n_o-1)}$$

follows an F-distribution with n_1-1 and n_o-1 degrees of freedom and we obtain a prediction interval for S_1 as;

$$\Pr\{F(n_1-1,n_0-1;(1-r)/2) < \frac{S_1^2}{S_0^2} < F(n_1-1,n_0-1;(1+r)/2)\} = r$$

or equivalently,

$$\Pr\{S_0F(n_1-1,n_0-1;(1-r)/2)^{\frac{1}{2}} < S_1 < S_0F(n_1-1,n_0-1;(1+r)/2)^{\frac{1}{2}}\} = r \tag{3.4}$$

where r is the chosen confidence coefficient and $F(n_1-1,n_0-1;(1-r)/2)$ and $F(n_1-1,n_0-1;(1+r)/2)$ are the appropriate percentage points of the F distribution with n_1-1 and n_0-1 degrees of freedom. This yields the following two-sided 100r% prediction interval to contain the standard deviation S_1 of n_1 future observations;

$$(S_0[\frac{1}{F(n_0-1,n_1-1;(1+r)/2)}]^{\frac{1}{2}}, F(n_1-1,n_0-1;(1+r)/2)^{\frac{1}{2}}S_0) \tag{3.5}$$

This prediction interval for S_1 is exactly the same as the one obtained by Hahn [5] for the independent case.

B. A SIMULTANEOUS PREDICTION INTERVAL TO CONTAIN THE STANDARD DEVIATION OF EACH OF k FUTURE SAMPLES

As in the previous section, let $X_{01},X_{02},X_{03}\cdots,X_{0n_0}$ be the values of a given random sample and let $X_{11},X_{12},X_{13},\ldots,X_{1n_1},X_{21},X_{22},X_{23}\cdots,X_{2n_2},X_{31},X_{32},X_{33},\ldots,X_{3n_3},$ $\ldots,X_{K1},X_{K2},X_{K3},\ldots,X_{Kn_k}$ be the values of K sets of future samples from a normal distribution with unknown mean μ and unknown standard deviation σ.

Let

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$$

$$S_i^{\,2} = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2,$$

where $i = 0,1,2,3,\ldots,k$ and let

$$N = \sum_{i=0}^{K} n_i.$$

Also let

$$\bar{X} = \frac{1}{N} \sum_{i=0}^{K} \sum_{j=1}^{n_i} X_{ij}$$

and

$$S^2 = \frac{1}{N} \sum_{i=0}^{K} \sum_{j=1}^{n_i} (X_{ij} - \bar{X})^2$$

be the mean and variance of the pooled sample of $N = \sum_{i=0}^{K}$ observations.

The sum of squares NS^2 can be partitioned as

$$NS^2 = \sum_{i=0}^{K} \sum_{j=1}^{n_i} (X_{ij}-\bar{X})^2 = \sum_{i=0}^{K} \sum_{j=1}^{n_i} (X_{ij}-\bar{X}_i+\bar{X}_i-\bar{X})^2$$

$$= \sum_{i=0}^{K} [\sum_{j=1}^{n_i} (X_{ij}-\bar{X}_i)^2 + n_i(\bar{X}_i-\bar{X})^2]$$

$$= n_o S_o^{\,2} + n_1 S_1^{\,2} + n_2 S_2^{\,2} + \ldots + n_K S_K^{\,2} + \sum_{i=0}^{K} n_i(\bar{X}_i-\bar{X})^2 \tag{3.6}$$

It follows that $(N-1)S^2/\sigma^2$ has a Chi-square distribution with N-1 degrees of freedom and $(n_i-1)S_i^{\,2}/\sigma^2$, $i=0,1,2,3,\ldots,K$, have Chi-square distributions with n_i-1 degrees of freedom respectively, and the last term of (3.6) is non-negative. Applying Hogg and Craig theorem (theorem 5) we can conclude that the last term of (3.6) also has a Chi-square distribution with $K[(N-1)-\sum_{i=0}^{K}(n_i-1) = (N-1)-(N-(K+1))=K]$ degrees of freedom and that all the sums of squares on the right hand side of equation (3.6) are mutually independent. Expressing these sums of squares as quadratic forms we can write equation (3.6) as;

$$\underline{X}'\underline{B}\underline{X} = \underline{X}'\underline{B}_0\underline{X} + \underline{X}'\underline{B}_1\underline{X} + \underline{X}'\underline{B}_2\underline{X} + \ldots + \underline{X}'\underline{B}_k\underline{X} + \underline{X}'\underline{B}_{k+1}\underline{X} \qquad (3.7)$$

where

$$\underline{B} = \underset{N\text{x}N}{\underline{I}} - N^{-1}\underset{N\text{x}N}{\underline{E}} \quad \text{and} \quad \underline{B}_i = \underset{n_i\text{x}n_i}{\underline{I}} - n_i^{-1}\underset{n_i\text{x}n_i}{\underline{E}}, \quad i=0,1,2,\ldots,K,K+1,$$

are idempotent matrices (see theorem 1).

Now, suppose $\underset{N\text{x}1}{\underline{X}}$ is a random vector having a multivariate normal distribution with mean $\underset{N\text{x}1}{\underline{\mu}} = (\mu,\mu,\mu,\ldots,\mu)'$ and covariance matrix $\underset{N\text{x}N}{V}$ which has the form (3.3).

The partition of NS^2 given in equations (3.6) and (3.7) are valid for this case also. Thus, we know $\underline{B}$ and $\underline{B}_i$, $i=0,1,2,3,\ldots,K$, are idempotent matrices and $\underline{B} = \underset{N\text{x}N}{\underline{I}} - N^{-1}\underset{N\text{x}N}{\underline{E}}$

$\underline{B}_i = \underline{I}_{n_i x n_i} - n_i^{-1} \underline{E}_{n_i x n_i}$. So, the conditions of Baldessari theorem (theorem 6) are all satisfied. Therefore, $\underline{X}'\underline{BX}/\alpha$ has a Chi-square distribution with N-1 degrees of freedom, $\underline{X}'\underline{B}_i\underline{X}/\alpha$, i=0,1,2,3,...,K, have Chi-square distributions with n_i-1 degrees of freedom and $\underline{X}'\underline{B}_{k+1}\underline{X}/\alpha$ has a Chi-square distribution with K degrees of freedom. Further the k+2 sums of squares on the right hand side of (3.7) are mutually independent. Thus, each of the random variables

$$\frac{\underline{X}'\underline{B}_i\underline{X}}{\alpha} \Bigg/ \frac{\underline{X}'\underline{B}_o\underline{X}}{\alpha} = \frac{(n_i-1)S_i^2}{\alpha(n_i-1)} \Bigg/ \frac{(n_o-1)S_o^2}{\alpha(n_o-1)} = \frac{S_i^2}{S_o^2}$$

where i = 1,2,3,...,k, follows an F distribution with n_i-1 and n_o-1 degrees of freedom.

Now, assume $n_1 = n_2 = n_3 = \ldots = n_k = m$. Then, S_i^2/S_o^2, i=1,2,3,...,K, has an F distribution with m-1 and n_o-1 degrees of freedom.

Define the random variables

$$W_L(K,m-1,n_o-1) = \max_i \frac{S_i^2}{S_o^2}$$

and

$$W_S(K,m-1,n_o-1) = \min_i \frac{S_i^2}{S_o^2}, \quad i=1,2,3,\ldots,K.$$

The distributions of $W_L(K,m-1,n_o-1)$ and $W_S(K,m-1,n_o-1)$ are known as the studentized largest and studentized smallest Chi-square distributions, respectively. The upper percentage point $D_U(K,m-1,n_o-1;r)$ of $W_L(K,m-1,n_o-1)$ and the lower percentage point $D_L(K,m-1,n_o-1;1-r)$ of $W_S(K,m-1,n_o-1)$ were tabulated by Armitage, J.V. and Krishnaiah, P.R. and are available in [1].

Then,

$$\Pr\{W_L(K,m-1,n_o-1) \leq D_U(K,m-1,n_o-1;r)\}$$

$$= \Pr\{\max_i \frac{S_i^2}{S_o^2} \leq D_U(K,m-1,n_o-1;r)\} = r$$

and

$$\Pr\{\max_i S_i^2 \leq D_U(K,m-1,n_o-1;r)S_o^2\} = r.$$

Thus, an upper 100r% simultaneous prediction limit to exceed the standard deviations of all k future samples each of size m is

$$S_o D_U(K,m-1,n_o-1;r)^{\frac{1}{2}} \tag{3.8}$$

Similarly, a lower 100(1-r)% simultaneous prediction limit to be exceeded by the standard deviations of each k future samples of size m is

$$S_o D_L(K,m-1,n_o-1,1-r)^{\frac{1}{2}} \tag{3.9}$$

This result is also the same as the one Hahn [5] obtained for independent samples.

C. NUMERICAL EXAMPLES

Suppose a gun is selected at random and fired $n_o = 6$ times and the resulting miss distances from a target are measured. Let $S_o = 1.00$ be the standard deviation of these observations. A prediction interval for the standard deviation S_1 of $n_1 = 10$ future attempts is desired.

Then, a two-sided 95% prediction interval to contain the standard deviation S_1 for a single future sample of 10 observations is obtained as follows;

For $n_1 = 10$, $n_o = 6$ and $r = 0.95$, $F(9,5;.975) = 6.68$ and $F(5,9;0.975) = 4.48$ and $S_oF(9,5;0.975)^{\frac{1}{2}} = (1.00)(6.68)^{\frac{1}{2}} = 2.584$ $S_oF(5,9;0.975)^{-\frac{1}{2}} = (1.00)(4.48)^{-\frac{1}{2}} = 0.472$.

Substituting the above in equation (3.5) the required prediction interval for S_1 is (0.472,2.584). Next, an upper 95% simultaneous limit to exceed the standard deviation of all 3 future samples of size 10 is;

For $m = 10$, $n_o = 6$, $K = 3$ and $r = 0.95$, $D_U(3,9,5;0.95) = 6.41$ and $D_U(3,9,5;0.95)^{\frac{1}{2}}S = (6.41)^{\frac{1}{2}}(1.00) = 2.762$ (see table 28 page 41 [1]).

IV. PREDICTION INTERVALS FOR THE ADDITIONAL OBSERVATIONS IN A FUTURE SAMPLE – CORRELATED CASE

A prediction interval to contain a single future observation and a simultaneous interval to contain each of k additional observations of a random sample from a normal distribution with mean μ and variance σ^2 were obtained by Hahn [5]. In this chapter we extend these results to the case where the samples are correlated and the covariance matrix has the form defined in (3.3). In section A a prediction interval to contain a single additional observation based on correlated observations is obtained, and section B deals with the construction of simultaneous prediction intervals to contain k additional correlated observations. Numerical examples are given in section C.

A. A PREDICTION INTERVAL FOR A SINGLE FUTURE OBSERVATION

Let $X_1, X_2, X_3, \ldots, X_n$ be independent and have identical normal distribution with unknown mean μ and unknown standard deviation σ. It is required to construct a prediction interval for an additional observation X_{n+1} based on the given n samples.

Let

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$S_n^{\ 2} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$\bar{X}_{n+1} = \frac{1}{n+1} \sum_{i=1}^{n+1} X_i$$

$$S_{n+1}^2 = \frac{1}{n+1} \sum_{i=1}^{n+1} (X_i - \bar{X}_{n+1})^2$$

Since

$$X_{n+1} = (n+1)\bar{X}_{n+1} - n\bar{X}_n \ , \quad \rightarrow \bar{X}_{n+1} = \frac{1}{n+1}\{X_{n+1} + n\bar{X}_n\}$$

the sum of squares $(n+1)S_{n+1}^2$ can be partitioned as follows:

$$(n+1)S_{n+1}^2 = \sum_{i=1}^{n+1} (X_i-\bar{X}_{n+1})^2 = \sum_{i=1}^{n} (X_i-\bar{X}_{n+1})^2 + (X_{n+1}-\bar{X}_{n+1})^2$$

$$= \sum_{i=1}^{n} (\underbrace{X_i-\bar{X}_n}_{A}+\underbrace{\bar{X}_n-\bar{X}_{n+1}}_{B})^2 + (X_{n+1}-\bar{X}_{n+1})^2$$

$$\sum_{1}^{n} 2(X_i-\bar{X}_n)(\bar{X}_n-\bar{X}_{n+1}) + (\bar{X}_n-\bar{X}_{n+1})^2$$

$$= \sum_{i=1}^{n} (X_i-\bar{X}_n)^2 + n\{\bar{X}_n - \frac{1}{n+1}(X_{n+1}+n\bar{X}_n)\}^2+\{X_{n+1}- \frac{1}{n+1}(X_{n+1}+n\bar{X}_n)\}$$

$$= nS_n^{\ 2} + n(\frac{X_{n+1}-\bar{X}_n}{n+1})^2 + \{\frac{n(X_{n+1}-\bar{X}_n)}{n+1}\}^2$$

$$= nS_n^{\ 2} + \frac{n}{n+1} (X_{n+1} - \bar{X}_n)^2 \qquad (4.1)$$

Expressing the sum of squares in (4.1) as quadratic forms we get

$$\underline{X}'\underline{B}_o\underline{X} = \underline{X}'\underline{B}_1\underline{X} + \underline{X}'\underline{B}_2\underline{X} \qquad (4.2)$$

where $\underset{\sim}{B}_o$ and $\underset{\sim}{B}_1$ are idempotent matrices and

$$\underset{\sim}{B}_o = \underset{(n+1)x(n+1)}{\underset{\sim}{I}} - (n+1)^{-1}\underset{(n+1)x(n+1)}{\underset{\sim}{E}} \quad \text{and} \quad \underset{\sim}{B}_1 = \underset{nxn}{\underset{\sim}{I}} - n^{-1}\underset{nxn}{\underset{\sim}{E}}.$$

Since $(n+1)S^2_{n+1}/\sigma^2$ and $nS_n{}^2/\sigma^2$ have Chi-square distributions with n and n-1 degrees of freedom respectively and $n(X_{n+1} - \overline{X}_n)^2/n+1$ is non-negative, Hogg and Craig's theorem (theorem 5) applies to equation (4.1). Therefore, the two quadratic forms on the right hand side of (4.1) are mutually independent and $n(X_{n+1} - \overline{X}_n)^2/(n+1)\sigma^2$ has a Chi-square distribution with 1 degree of freedom. It also follows that the matrix $\underset{\sim}{B}_2$ in equation (4.2) is also idempotent.

Now, suppose $\underset{(n+1)x1}{\underset{\sim}{X}}$ is a vector random variable having a multivariate normal distribution with mean $\underset{(n+1)x1}{\underset{\sim}{\mu}} = (\mu,\mu,\mu,\ldots,\mu)'$ and covariance matrix $\underset{(n+1)x(n+1)}{\underset{\sim}{V}}$ which has the following structure.

$$\underset{(n+1)x(n+1)}{\underset{\sim}{V}} = \frac{1}{2}(\underset{(n+1)x(n+1)}{\underset{\sim}{H}} + \underset{(n+1)x(n+1)}{\underset{\sim}{H'}}) + \alpha(\underset{(n+1)x(n+1)}{\underset{\sim}{I}} - \underset{(n+1)x(n+1)}{\underset{\sim}{E}}) \quad (4.3)$$

where

$$\underset{(n+1)x(n+1)}{\underset{\sim}{H}} = \begin{pmatrix} h_1 & h_1 & h_1 & \cdots\cdots & h_1 \\ h_2 & h_2 & h_2 & \cdots\cdots & h_2 \\ h_3 & h_3 & h_3 & & h_3 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ h_{n+1} & h_{n+1} & h_{n+1} & \cdots\cdots & h_{n+1} \end{pmatrix},$$

σ be linear transf. of an n-dimensional vector space V into itself

$\sigma^2 = \sigma$ is said to be idempotent

All characteristic roots $= 0$ or 1.

$A^2 = A$.

$\underline{H}'$ is the transpose of $\underline{H}$, α and h_i, $i=1,2,3,\ldots,n+1$ are positive constant, $\underline{I}$ is an $(n+1)x(n+1)$ identity matrix and $\underline{E}$ is an $(n+1)x(n+1)$ matrix whose elements are all unity. Since the matrices $\underline{B}_0$, $\underline{B}_1$, and $\underline{B}_2$ in equation (4.2) are idempotent and

$$\underset{(n+1)x(n+1)}{\underline{B}_0} = \sum_{j=1}^{2} \underline{B}_j = \underset{(n+1)x(n+1)}{\underline{I}} - (n+1)^{-1} \underset{(n+1)x(n+1)}{\underline{E}} ,$$

we may apply the Baldessari theorem (Theorem 6) to equation (4.1) to conclude that the two quadratic forms on the right hand side of the equations have central Chi-square distribution with n-1 and 1 degree of freedom respectively and are mutually independent.

Thus, the random variable

$$\frac{\underline{X}'\underline{B}_2\underline{X}}{\alpha} \bigg/ \frac{\underline{X}'\underline{B}_1\underline{X}}{\alpha} = \frac{(\frac{n}{n+1})(X_{n+1} - \overline{X}_n)^2}{\alpha} \bigg/ \frac{(n-1)S_n{}^2}{\alpha(n-1)}$$

has an F distribution with 1 and n-1 degree of freedom. We obtain a prediction interval for X_{n+1} as

$$\Pr\{F(1,n-1;(1-r)/2) < (\tfrac{n}{n+1})(X_{n+1}-\overline{X}_n)^2 \big/ S_n{}^2 < F(1,n-1;(1+r)/2)\} = r$$

or equivalently

$$\Pr\{\overline{X}_n+S_n(1+\tfrac{1}{n})^{\frac{1}{2}}F(1,n-1;(1-r/2)^{\frac{1}{2}} < X_{n+1} <$$

$$< \overline{X}_n+S_n(1+\tfrac{1}{n})^{\frac{1}{2}}F(1,n-1;(1+r)/2)^{\frac{1}{2}}\} = r$$

where r is the chosen confidence coefficient and $F(1,n-1;(1+r)/2)$ and $F(1,n-1;(1+r)/2)$ are the appropriate percentage points of the F distribution with 1 and n-1 degree of freedom.

Now recall that $F(1,n-1;(1-r)/2)^{\frac{1}{2}} = t(n-1;(1-r)/2)$

and $$F(1,n-1;(1+r)/2)^{\frac{1}{2}} = t(n-1;(1+r)/2)$$

This yields the following two-sided prediction interval to contain the additional observation X_{n+1}:

$$\underline{(\overline{X}_n+S_n(1+\tfrac{1}{n})^{\frac{1}{2}}t(n-1;(1-r)/2),\overline{X}_n+S_n(1+\tfrac{1}{n})^{\frac{1}{2}}t(n-1;(1+r)/2))} \quad (4.4)$$

B. SIMULTANEOUS PREDICTION INTERVALS FOR k FUTURE OBSERVATIONS

Let $X_1,X_2,X_3,\ldots,X_n$ be the values of a given sample and $X_{n+1},X_{n+2},X_{n+3},\ldots,X_{n+k}$, the values of k future observations. We assume that the sample observations $X_1,X_2,X_3,\ldots,X_n$, $X_{n+1},X_{n+2},\ldots,X_{n+k}$ are correlated and have a multivariate normal distribution with mean $\underline{\mu} = (\mu,\mu,\mu,\ldots,\mu)'$ and covariance matrix $\underline{V}$ which has the form (4.3).

In order to construct a simultaneous prediction interval for $X_{n+1}, X_{n+2}, X_{n+3}, \ldots, X_{n+k}$ we first establish that

(i) $\sum_{i=1}^{n} \frac{(X_i - \overline{X}_n)^2}{\alpha} = \frac{nS^2}{\alpha}$ has a Chi-square distribution

with n-1 degree of freedom,

(ii) the vector variable $\underline{Z} = (Z_1, Z_2, Z_3, \ldots, Z_k)'$, where

$Z_i = X_{n+i} - \overline{X}_n$, has a multivariate normal distribution and

(iii) the vector variable $\underline{Z}$ and nS^2 are statistically independent.

If $nS^2 = \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$ is expressed as a quadratic form $\underline{X}'\underline{BX}$, where $\underline{X} = (X_1, X_2, X_3, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{n+k})'$, then a necessary and sufficient condition for $\underline{X}'\underline{BX}$ to have a Chi-square distribution is that $\underline{BV}$ is idempotent (see Theorem 2).

To show that $\underline{BV}$ is idempotent, let the matrices $\underline{H}$, $\underline{H}'$, and $\underline{E}$ (4.3) and the matrix $\underline{V}$ be partitioned as follows:

$$
\underline{H} = \left(\begin{array}{ccccc|ccc}
h_1 & h_1 & h_1 & \cdots & h_1 & h_1 & \cdots & h_1 \\
h_2 & h_2 & h_2 & \cdots & h_2 & h_2 & \cdots & h_2 \\
h_3 & h_3 & h_3 & \cdots & h_3 & h_3 & \cdots & h_3 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
h_n & h_n & h_n & \cdots & h_n & h_n & \cdots & h_n \\
\hline
h_{n+1} & h_{n+1} & h_{n+1} & \cdots & h_{n+1} & h_{n+1} & \cdots & h_{n+1} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
h_{n+k} & h_{n+k} & h_{n+k} & \cdots & h_{n+k} & h_{n+k} & \cdots & h_{n+k}
\end{array}\right)
$$

$$
\hphantom{\underline{H} = (}\underbrace{\hphantom{h_{n+k}\; h_{n+k}\; h_{n+k}\; \cdots\; h_{n+k}}}_{n}\;\underbrace{\hphantom{h_{n+k}\; \cdots\; h_{n+k}}}_{k}
$$

$$
= \left(\begin{array}{c|c}
\underset{n\text{x}n}{\underline{H}_1} & \underset{n\text{x}k}{\underline{H}_2} \\
\hline
\underset{k\text{x}n}{\underline{H}_3} & \underset{k\text{x}k}{\underline{H}_4}
\end{array}\right)
$$

$$
\underline{H'} = \left(\begin{array}{c|c}
\underset{n\text{x}n}{\underline{H}_1'} & \underset{n\text{x}k}{\underline{H}_3'} \\
\hline
\underset{k\text{x}n}{\underline{H}_2'} & \underset{k\text{x}k}{\underline{H}_4'}
\end{array}\right)
$$

$$
\underline{E} = \left(\begin{array}{cccccc|ccccc}
1 & 1 & 1 & \cdots & & 1 & 1 & \cdots & & & 1 \\
1 & 1 & 1 & \cdots & & 1 & 1 & \cdots & & & 1 \\
1 & 1 & 1 & \cdots & & 1 & 1 & \cdots & & & 1 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & & & & \vdots \\
1 & 1 & 1 & \cdots & & 1 & 1 & \cdots & & & 1 \\
\hline
1 & 1 & 1 & \cdots & & 1 & 1 & \cdots & & & 1 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & & & & \vdots \\
1 & 1 & 1 & \cdots & & 1 & 1 & \cdots & & & 1
\end{array}\right)
$$

$$
\underbrace{\hphantom{1 \quad 1 \quad 1 \quad \cdots \quad 1}}_{n} \qquad \underbrace{\hphantom{1 \quad \cdots \quad 1}}_{k}
$$

$$
= \left(\begin{array}{c|c}
\underset{n\mathrm{x}n}{\underline{E}_1} & \underset{n\mathrm{x}k}{\underline{E}_2} \\
\hline
\underset{k\mathrm{x}n}{\underline{E}_3} & \underset{k\mathrm{x}k}{\underline{E}_4}
\end{array}\right)
$$

and

$$
\underline{V} = \left(\begin{array}{c|c}
\underset{n\mathrm{x}n}{\underline{V}_1} & \underset{n\mathrm{x}k}{\underline{V}_2} \\
\hline
\underset{k\mathrm{x}n}{\underline{V}_3} & \underset{k\mathrm{x}k}{\underline{V}_4}
\end{array}\right)
$$

$$\underline{I} = \left(\begin{array}{c|c} \underset{n x n}{\underline{I}_1} & 0 \\ \hline 0 & \underset{k x k}{\underline{I}_4} \end{array}\right)$$

where $\underline{V}_1 = \frac{1}{2}(\underline{H}_1+\underline{H}_1{}') + \alpha(\underline{I}_1-\underline{E}_1)$

and $\underline{V}_2 = \frac{1}{2}(\underline{H}_2+\underline{H}_3{}') + \alpha(0-\underline{E}_2)$

We know

$$\underset{(nxk)x(n+k)}{\underline{B}} = \left(\begin{array}{c|c} \underset{n x n}{\underline{B}_1} & 0 \\ \hline 0 & 0 \end{array}\right)$$

where $\underset{n x n}{\underline{B}_1} = \frac{1}{\alpha}\left(\underset{n x n}{\underline{I}} - n^{-1} \underset{n x n}{\underline{E}} \right)$

and
$$\underline{BV} = \left(\begin{array}{c|c} \underline{B}_1 & 0 \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} \underline{V}_1 & \underline{V}_2 \\ \hline \underline{V}_3 & \underline{V}_4 \end{array}\right) = \left(\begin{array}{c|c} \underline{B}_1\underline{V}_1 & \underline{B}_1\underline{V}_2 \\ \hline 0 & 0 \end{array}\right) \tag{4.5}$$

In equation (4.5), $\underline{B}_1\underline{V}_1$ and $\underline{B}_1\underline{V}_2$ can be simplified as follows:

$$B_1V_1 = \frac{1}{\alpha}(\underline{I}_1 - n^{-1}\underline{E}_1)\{\tfrac{1}{2}(\underline{H}_1 + \underline{H}_1') + \alpha(\underline{I}_1 - \underline{E}_1)\}$$

$$= \frac{1}{\alpha}(\tfrac{1}{2}\underline{H}_1 + \tfrac{1}{2}\underline{H}_1' + \alpha\underline{I}_1 - \alpha\underline{E}_1 - \frac{1}{2n}\underline{E}_1\underline{H}_1 - \frac{1}{2n}\underline{E}_1\underline{H}_1'$$

$$- \frac{\alpha}{n}\underline{E}_1 I_1 + \frac{\alpha}{n}\underline{E}_1\underline{E}_1)$$

$$= \frac{1}{\alpha}(\tfrac{1}{2}\underline{H}_1 + \tfrac{1}{2}\underline{H}_1' + \alpha\underline{I}_1 - \alpha\underline{E}_1 - \frac{a}{2n}\underline{E}_1 - \frac{n}{2n}\underline{H}_1'$$

$$- \frac{\alpha}{n}\underline{E}_1 + \frac{\alpha}{n}n\underline{E}_1)$$

$$= (\underline{I}_1 - \frac{1}{n}\,\underline{E}_1) + \frac{1}{2\alpha}\,(\underline{H}_1 - \frac{a}{n}\,\underline{E})$$

where $a = \sum_{i=1}^{n} h_i$

and $$\underline{B}_1\underline{V}_2 = \frac{1}{\alpha}(\underline{I}_1 - n^{-1}\underline{E}_1)\{\tfrac{1}{2}(\underline{H}_2 + \underline{H}_3') + \alpha(0 - \underline{E}_2)\}$$

$$= \frac{1}{\alpha}(\tfrac{1}{2}\underline{H}_2 + \tfrac{1}{2}\underline{H}_3' - \alpha\underline{E}_2 - \frac{1}{2n}\underline{E}_1\underline{H}_2 - \frac{1}{2n}\underline{E}_1\underline{H}_3' + \frac{\alpha}{n}\underline{E}_1\underline{E}_2)$$

$$= \frac{1}{\alpha}(\tfrac{1}{2}\underline{H}_2 + \tfrac{1}{2}\underline{H}_3' - \alpha\underline{E}_2 - \frac{a}{2n}\underline{E}_2 - \frac{n}{2n}\underline{H}_3' + \frac{\alpha}{n}n\underline{E}_2)$$

$$= \frac{1}{2\alpha}(\underline{H}_2 - \frac{a}{n}\underline{E}_2)$$

Thus

$$\underline{BV} = \left(\begin{array}{c|c} \underline{B}_1\underline{V}_1 & \underline{B}_1\underline{V}_2 \\ \hline 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} \underline{I}_1 - n^{-1}\underline{E}_1 + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1) & \frac{1}{2\alpha}(\underline{H}_2 - \frac{a}{n}\underline{E}_2) \\ \hline 0 & 0 \end{array}\right)$$

and

$$(\underline{BV})(\underline{BV}) = \left(\begin{array}{c|c} \underline{B}_1\underline{V}_1 & \underline{B}_1\underline{V}_2 \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} \underline{B}_1\underline{V}_1 & \underline{B}_1\underline{V}_2 \\ \hline 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} (\underline{B}_1\underline{V}_1)^2 & (\underline{B}_1\underline{V}_1)(\underline{B}_1\underline{V}_2) \\ \hline 0 & 0 \end{array}\right) \qquad (4.6)$$

But

$$(\underline{B}_1\underline{V}_1)^2 = \{(\underline{I}_1 - n^{-1}\underline{E}_1) + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)\}^2$$

$$= (\underline{I}_1 - n^{-1}\underline{E}_1)^2 + (\underline{I}_1 - n^{-1}\underline{E}_1)\frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)$$

$$+ \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)(\underline{I}_1 - n^{-1}\underline{E}_1) + \{\frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)\}^2$$

$$= \underline{I}_1 - \frac{1}{n}\underline{E}_1 + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1 - \frac{1}{n}\underline{E}_1\underline{H}_1 + \frac{a}{n^2}\underline{E}_1\underline{E}_1)$$

$$+ \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1 - \frac{1}{n}\underline{H}_1\underline{E}_1 + \frac{a}{n^2}\underline{E}_1\underline{E}_1)$$

$$+ \frac{1}{4\alpha^2}(\underline{H}_1\underline{H}_1 - \frac{a}{n}\underline{H}_1\underline{E}_1 - \frac{a}{n}\underline{E}_1\underline{H}_1 + \frac{a^2}{n^2}\underline{E}_1\underline{E}_1)$$

$$= \underline{I}_1 - \frac{1}{n}\underline{E}_1 + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1 - \frac{a}{n}\underline{E}_1 + \frac{a}{n^2}n\underline{E}_1)$$

$$+ \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1 - \frac{n}{n}\underline{H}_1 + \frac{a}{n^2}n\underline{E}_1)$$

$$+ \frac{1}{4\alpha^2}(a\underline{H}_1 - \frac{a}{n}n\underline{H}_1 - \frac{a}{n}a\underline{E}_1 + \frac{a^2}{n^2}n\underline{E}_1)$$

$$= \underline{I}_1 - \frac{1}{n}\underline{E}_1 + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)$$

and $$(\underline{B}_1\underline{V}_1)(\underline{B}_1\underline{V}_2) = \{(\underline{I}_1 - \frac{1}{n}\underline{E}_1) + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)\}$$

$$\times\{\frac{1}{2\alpha}(\underline{H}_2 - \frac{a}{n}\underline{E}_2)\}$$

$$= \frac{1}{2\alpha}\{(\underline{I}_1 - \frac{1}{n}\underline{E}_1)\underline{H}_2 + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)\underline{H}_2$$

$$- \frac{a}{n}(\underline{I}_1 - \frac{1}{n}\underline{E}_1)\underline{E}_2 - \frac{1}{2\alpha n}(\underline{H}_1 - \frac{a}{n}\underline{E}_1)\underline{E}_2\}$$

$$= \frac{1}{2\alpha}\{\underline{H}_2 - \frac{a}{n}\underline{E}_2 + \frac{a}{2\alpha}\underline{H}_2 - \frac{a^2}{2\alpha n}\underline{E}_2 - \frac{a}{n}\underline{E}_2$$

$$+ \frac{an}{n^2}\underline{E}_2 - \frac{an}{2\alpha n}\underline{H}_2 + \frac{a^2 n}{2\alpha n^2}\underline{E}_2\}$$

$$= \frac{1}{2\alpha}(\underline{H}_2 - \frac{a}{n}\underline{E}_2)$$

Therefore

$$(\underline{BV})(\underline{BV}) = \left(\begin{array}{c|c} \underline{I}_1 - \frac{1}{n}\underline{E}_1 + \frac{1}{2\alpha}(\underline{H}_1 - \frac{a}{n}\underline{E}_1) & \frac{1}{2\alpha}(\underline{H}_2 - \frac{a}{n}\underline{E}_2) \\ \hline 0 & 0 \end{array}\right)$$

$$= \underline{BV}$$

Since $\underline{BV}$ is idempotent, $\underline{X}'\underline{BX}/\alpha$ has a chi-square distribution with n-1 degree of freedom.

Next, $\underline{Z} = (Z_1, Z_2, Z_3, \ldots, Z_k)'$ can be expressed as

$$\underset{k\times 1}{\underline{Z}} = \underset{k\times(n+k)}{\underline{C}'} \; \underset{(n+k)\times 1}{\underline{X}}$$

where $\underline{C}' = \left(\frac{-1}{n} \underset{k\times n}{\underline{E}} \, , \, \underset{k\times k}{\underline{I}} \right)$

$\underline{Z}$ has a multivariate normal distribution with mean $\underline{C}'\underline{\mu} = \underline{0}$ and covariance matrix $\underline{C}'\underline{VC}$ as shown below:

$$\underline{C}'\underline{\mu} = \left(\frac{-1}{n} \underset{k\times n}{\underline{E}} \, , \, \underset{k\times k}{\underline{I}} \right) \underset{(k+n)\times 1}{\underline{\mu}} = \mu\left(\frac{-1}{n} n + 1\right) = 0$$

$$\underline{C}'\underline{V} = \left(\frac{-1}{n} \underset{k\times n}{\underline{E}} \;,\; \underset{k\times k}{\underline{I}} \right) \left[\frac{1}{2}\left\{ \left(\begin{array}{c|c} \underline{H}_1 & \underline{H}_2 \\ \hline \underline{H}_3 & \underline{H}_4 \end{array} \right) + \left(\begin{array}{c|c} \underline{H}_1' & \underline{H}_3' \\ \hline \underline{H}_2' & \underline{H}_4' \end{array} \right) \right\} \right.$$

$$\left. + \alpha \left(\begin{array}{c|c} \underline{I}_1 & 0 \\ \hline 0 & \underline{I}_4 \end{array} \right) - \alpha \left(\begin{array}{c|c} \underline{E}_1 & \underline{E}_2 \\ \hline \underline{E}_3 & \underline{E}_4 \end{array} \right) \right]$$

$$= \frac{1}{2} \underset{1\times(n+k)}{\begin{pmatrix} (\frac{-a}{n} + h_{n+1})\underline{E} \\ (\frac{-a}{n} + h_{n+2})\underline{E} \\ \\ (\frac{-a}{n} + h_{n+k})\underline{E} \end{pmatrix}}$$

$$+ \frac{1}{2} \left\{ (\frac{-n}{n}h_1 + h_1)\underset{k\times 1}{\underline{E}} \;,\; (\frac{-n}{n}h_2 + h_2)\underset{k\times 1}{\underline{E}} \;, \ldots ,\; (\frac{-n}{n}h_{n+k} + h_{n+k})\underset{k\times 1}{\underline{E}} \right\}$$

$$+ \alpha\left(\frac{-1}{n} \underset{k\times n}{E} \;,\; \underset{k\times k}{\underline{I}} \right) - \alpha\left(\frac{-1}{n}\, n + 1 \right) \underset{k\times n+k}{\underline{E}}$$

$$= \frac{1}{2}\begin{pmatrix} (\frac{-a}{n} + h_{n+1})\underline{E} \\ (\frac{-a}{n} + h_{n+2})\underline{E} \\ \vdots \\ (\frac{-a}{n} + h_{n+k})\underline{E} \end{pmatrix}_{1xn+k} + \alpha\left(\frac{-1}{n} \underset{kxn}{\underline{E}} , \underset{kxk}{\underline{I}} \right) \tag{4.7}$$

$$\underline{C}'\underline{VC} = \left\{ \frac{1}{2}\begin{pmatrix} (\frac{-a}{n} + h_{n+1})\underline{E} \\ (\frac{-a}{n} + h_{n+2})\underline{E} \\ \vdots \\ (\frac{-a}{n} + h_{n+k})\underline{E} \end{pmatrix}_{1x(k+n)} + \alpha\left(\frac{-1}{n} \underset{kxn}{\underline{E}} , \underset{kxk}{\underline{I}} \right) \right\}$$

$$\times \begin{pmatrix} \frac{-1}{n} \underset{nxk}{\underline{E}} \\ , \\ \underset{kxk}{\underline{I}} \end{pmatrix}$$

$$= \begin{pmatrix} [(\frac{-a}{n}+h_{n+1})(\frac{-n}{n}) + (\frac{-a}{n}+h_{n+1})]\underline{E} \\ [(\frac{-a}{n}+h_{n+2})(\frac{-n}{n}) + (\frac{-a}{n}+h_{n+2})]\underline{E} \\ \vdots \\ [(\frac{-a}{n}+h_{n+k})(\frac{-n}{n}) + (\frac{-a}{n}+h_{n+k})]\underline{E} \end{pmatrix}_{1xk}$$

$$+ \alpha\left(\frac{1}{n^2} \underset{kxk}{\underline{E}} \underset{kxk}{\underline{E}} + \underset{kxk}{\underline{I}} \right)$$

$$= \alpha\left(\frac{1}{n} \underset{k\times k}{\underline{E}} + \underset{k\times k}{\underline{I}} \right)$$

$$= \alpha \begin{pmatrix} \frac{1}{n}+1 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n}+1 & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n}+1 \end{pmatrix}$$

Therefore, $\underline{Z} \sim N(\underline{0}, \alpha(\frac{1}{n}\underline{E} + \underline{I}))$.

Also, $\underline{Z} = \underline{C}'\underline{X}$ and $\underline{X}'\underline{BX}$ are statistically independent since (see Theorem 4)

$$\underline{C}'\underline{VB} = \left\{ \frac{1}{2} \begin{pmatrix} (\frac{-a}{n}+h_{n+1}) & \underline{E} \\ (\frac{-a}{n}+h_{n+2}) & \underline{E} \\ \vdots & \\ (\frac{-a}{n}+h_{n+k}) & \underset{1\times(k+n)}{\underline{E}} \end{pmatrix} + \alpha\left(\frac{-1}{n} \underset{k\times n}{\underline{E}} , \underset{k\times k}{\underline{I}} \right) \right\}$$

$$\times \left(\begin{array}{c|c} \underset{n\times n}{\underline{I}} - n^{-1} \underset{n\times n}{\underline{E}} & 0 \\ \hline 0 & 0 \end{array} \right)$$

$$= \frac{1}{2}\begin{pmatrix} (\frac{-a}{n}+h_{n+1}) & \underline{E} \\ (\frac{-a}{n}+h_{n+2}) & \underline{E} \\ \vdots & \\ (\frac{-a}{n}+h_{n+k}) & \underline{E} \end{pmatrix}_{1xn} - \frac{\alpha}{n}\underset{kxn}{\underline{E}}$$

$$- \frac{1}{2n}\begin{pmatrix} (\frac{-a}{n}+h_{n+1})n\underline{E} \\ (\frac{-a}{n}+h_{n+2})n\underline{E} \\ \vdots \\ (\frac{-a}{n}+h_{n+k})n\underline{E} \end{pmatrix}_{1xn} + \frac{\alpha}{n^2}\, n\, \underset{kxn}{\underline{E}}$$

$$= 0$$

Thus, each Z_i, $i=1,2,3,\ldots,k$, is normally distributed with mean 0 and variance $\alpha(1 + 1/n)$ and is independent of S^2.

Let Z_i', $i=1,2,3,\ldots,k$, be the standardized variables defined by

$$Z_i' = \frac{Z_i - 0}{\{\alpha(1 + \frac{1}{n})\}^{\frac{1}{2}}} = \frac{X_{n+i} - \overline{X}}{\{\alpha(1 + \frac{1}{n})\}^{\frac{1}{2}}}$$

the variables

$$T_i = \frac{Z_i'}{\sqrt{\frac{(n-1)S^2}{\alpha(n-1)}}} = \frac{X_{n+i} - \overline{X}}{S(1 + \frac{1}{n})^{\frac{1}{2}}} \qquad i=1,2,3,\ldots,k\ .$$

are jointly distributed according to the multivariate generalization of the Student t-distribution with n-1 degree of freedom and correlation matrix Σ defined by

$$\Sigma = \begin{pmatrix} 1 & \frac{1}{n+1} & \frac{1}{n+1} & \cdots\cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & 1 & \frac{1}{n+1} & \cdots\cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+1} & 1 & \cdots\cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & & \cdots\cdots & 1 \end{pmatrix}$$

To find a two-sided 100r% simultaneous prediction interval to contain each of k additional observations, let U be such that

$$r = \int_{-U}^{U} \int_{-U}^{U} \cdots \int_{-U}^{U} f_{T_1,T_2,T_3,\ldots,T_k}(t_1,t_2,t_3,\ldots t_k;n-1,\Sigma)\, dt_1 dt_2 \ldots dt_k \tag{4.8}$$

Then

$$\Pr\left\{-U < \frac{X_{n+1} - \bar{X}}{S(1+\frac{1}{n})^{\frac{1}{2}}} < U \text{ and } \ldots, \text{ and } -U < \frac{X_{n+k} - \bar{X}}{S(1+\frac{1}{n})^{\frac{1}{2}}} < U\right\} = r$$

The resulting 100r% simultaneous prediction interval to contain the values $X_{n+1}, X_{n+2}, X_{n+3}, \ldots, X_{n+k}$ of all k future observations is

$$\bar{X} \pm U(1 + \frac{1}{n})^{\frac{1}{2}} S \quad . \tag{4.9}$$

For selected values of r, the values of U to satisfy the equation (4.8) were tabulated by Hahn and are available in [4].

C. NUMERICAL EXAMPLES

Based upon a random sample of observations from a normal distribution whose mean and standard deviation are unknown, the following data is obtained.

51.4, 49.5, 48.7, 49.3 and 51.6

From the data, the sample mean $\overline{X}$ and sample standard deviation S are calculated as

$$\overline{X} = (51.4 + 49.5 + 48.7 + 49.3 + 51.6)/5 = 50.10$$

and
$$S^2 = \{(51.4-50.1)^2 + (49.5-50.1)^2 + (48.7-50.1)^2 + (49.3-50.1)^2 + (51.6-50.1)^2\}/5$$

$$= 6.9/5 = 1.38$$

$$S = 1.175$$

Then, a two-sided prediction interval to contain a single future observation X_{n+1} with 95% probability is (see equation (4.4):

For n=5, r=0.95, from the Student's t-tables $t(4,0.975) = 2.776$. Substituting the observed values in (4.4) a 95% prediction interval for X_{n+1} a future observation is given by

$$(46.527,\ 53.673)$$

Next, a two-sided 95% simultaneous prediction interval to contain each of 10 future observations is obtained using equation (4.9):

For k=10, n=5 and r=0.95 from the tables in [4] $U(1+\frac{1}{n})^{\frac{1}{2}} = 5.23$. Thus $\overline{X} \pm U(1+\frac{1}{n})^{\frac{1}{2}}S = 50.1 \pm 5.23(1.175)$ and the required prediction interval is given by

$$(43.855,\ 56.145)$$

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